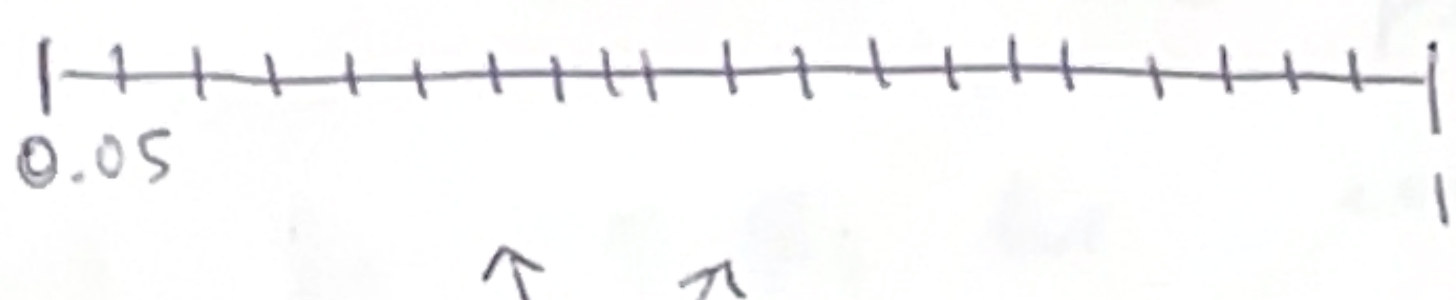
Bayesian hierarchical regression models =
mixed effects models

From Haigis et al 2004:

21 mice

intestine :



divided into 20 equally spaced regions

count the # of tumors in each region for each mouse.

In regression, recall we're always interested in $E\,Y|X$.

* From figure, tumor counts look more similar within a mouse than btwn mice.

Let $\boxed{Y_{x,j}}$ be mouse j's tumor count at location x.

$$Y_{x,j} \sim \text{Poisson}\left(\theta_j(x)\right) \qquad p(Y_{x,j}|\theta_j(x)) = \frac{\theta_j(x)^{y_{x,j}} e^{-\theta_j(x)}}{y_{x,j}!}$$

$Y_{x,j} \sim \text{Poisson}(\theta_j(x))$

Data

$\mathbb{E}\, Y_{x,j} \mid x = \theta_j(x)$

$\log(\theta_j(x)) = X\beta = \beta_{1,j} + \beta_{2,j}x + \beta_{3,j}x^2 + \beta_{4,j}x^3 + \beta_{5,j}x^4$

<u>unknowns</u>: βs 5 βs for each mouse!

105 unknowns.

$$\begin{tikzcd}
 & \theta, \Sigma \arrow[dl] \arrow[dr] & \\
\vec{\beta}_1 \arrow[d] & \cdots & \vec{\beta}_{21} \arrow[d] \\
\vec{Y}_1 & \cdots & \vec{Y}_{21}
\end{tikzcd}$$

$\vec{\beta}_j \overset{iid}{\sim} MVN(\theta, \Sigma)$ ← btwn group sampling model.

This hierarchical regression model <u>is</u> a mixed effects model.

$\beta_j = \theta + \gamma_j \qquad \gamma_j \sim MVN(0, \Sigma)$

θ: fixed effect

γ_j: random effect

$X_j\beta = X_j(\theta + \gamma_j) = \theta_1 + \gamma_{1j} + \theta_2 X_j + \gamma_{2j}X_j + \theta_3 X_j^2 + \gamma_{3j}X_j^2 + \cdots$

Aside ex: General random effects model

$$Y_{ij} = \underset{\substack{\uparrow \\ \text{fixed} \\ \text{effects}}}{\theta^T} x_{ij} + \underset{\substack{\uparrow \\ \text{random} \\ \text{effects}}}{\gamma_j^T} z_{ij} + \varepsilon_{ij}$$

Putting it all together,

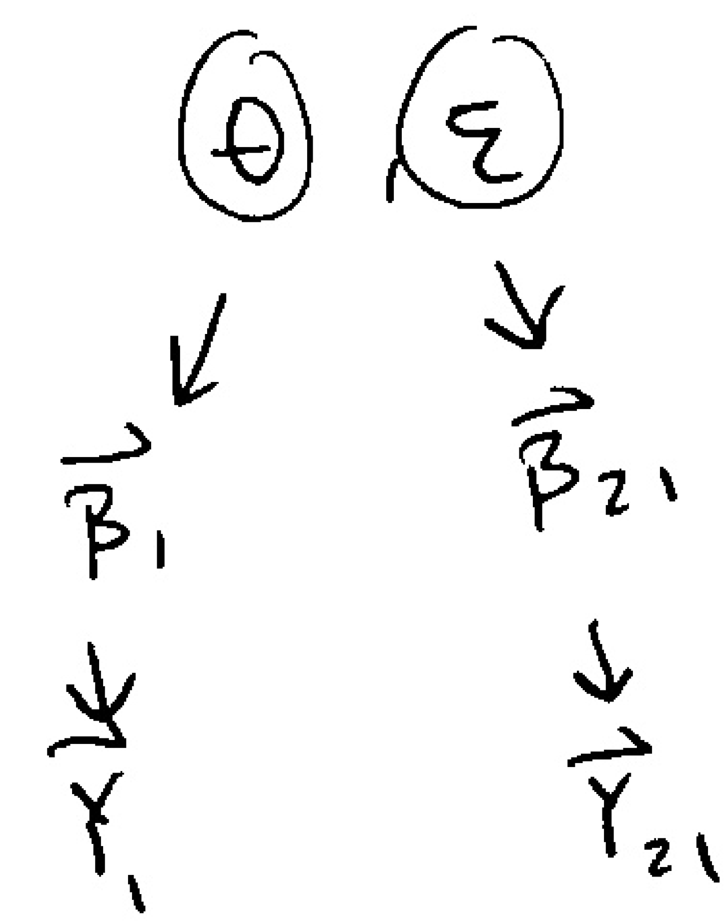
$$Y_{x,j} \mid x, \beta \sim \text{Poisson}(\theta_j(x))$$

$$\log \theta_j(x) = \beta_{1,j} + \beta_{2,j}x + \cdots + \beta_{5,j}x^4$$

$$\beta_j \mid \theta, \Sigma \sim MVN(\theta, \Sigma)$$

$$\theta \sim MVN(\underset{5\times 1}{\mu}, \underset{5\times 5}{\Lambda_0})$$

$$\Sigma \sim \text{inv-Wishart}(\underset{1\times 1}{n_0}, \underset{5\times 5}{S_0})$$



how to choose prior parameters?

one option : Unit info.

Quiz (1) Describe in words or pseudo-code an MCMC procedure to sample all unknowns.

$$p(\beta, \theta, \Sigma \mid Y, X) \propto \underbrace{p(y \mid \beta, \theta, \Sigma, X)\, p(\beta, \theta, \Sigma \mid X)}_{p(y \mid \beta, X)\, p(\beta \mid \theta, \Sigma)\, p(\theta)\, p(\Sigma)}$$

To facilitate Gibbs sampling, we need full cond'l post.

$$\underbrace{p(\theta \mid \cdot)}_{MVN} \propto \underbrace{p(\beta \mid \theta, \Sigma)}_{MVN}\, \underbrace{p(\theta)}_{MVN}$$

$$\underbrace{p(\Sigma \mid \cdot)}_{\text{inv-wish}} \propto p(\beta \mid \theta, \Sigma)\, p(\Sigma)$$

$$p(\beta \mid \cdot) \propto p(y \mid \beta, X)\, \underbrace{p(\beta \mid \theta, \Sigma)}_{MVN}$$

Recall $Y_{x,j} \sim \text{Poisson}(\theta_j(x))$

$$p(y \mid \beta, X) = \prod_{j=1}^{21} \frac{\theta_j(x)^{y_{x,j}} e^{-\theta_j(x)}}{y_{x,j}!}$$

$$\prod_{j=1}^{21} \frac{\left(e^{x\beta_j}\right)^{y_{x,j}} e^{-e^{x\beta_j}}}{y_{x,j}!}$$

can't identify full cond'l for $\beta \mid -$

$\Rightarrow$ Metropolis algo.

initialize $\beta^{(0)}, \cancel{\theta^{(0)}}$

for (s in 1:S){

Sample $\Sigma \sim$ inv-Wishart(_ , _)

sample $\theta \sim$ MVN(_ , _)

propose $\beta^* \sim MVN(\beta^{(s)}, S_0)$

accept/reject w/ prob. min(1, r)

$$r = \frac{p(\beta^* | \cdot)}{p(\beta^{(s)} | \cdot)}$$

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