Ex 1. Derive Bayes' rule:

$$p(H_i \mid X) = \frac{p(H_i, X)}{p(X)} \quad \text{by } P3$$

$$= \frac{p(X \mid H_i)\, p(H_i)}{p(X)} \quad \text{by } P3$$

$$= \frac{p(X \mid H_i)\, p(H_i)}{\sum_k p(X \mid H_k)\, p(H_k)} \quad \text{by rule of marginal prob.}$$

Ex 2: show $F \perp G \mid H \Rightarrow p(F \mid H, G) = p(F \mid H)$

(†) $p(F, G \mid H) = p(F \mid H)\, p(G \mid H)$ by definition

but also

(††) $p(F, G \mid H) = p(F \mid G, H)\, p(G \mid H)$ by P3

matching up (†) & (††):

$p(F \mid G, H)\, \cancel{p(G \mid H)} = p(F \mid H)\, \cancel{p(G \mid H)}$

Support: set of values a r.v. can take.

$X \sim \text{binomial}(n, \theta)$

$X \in \{0, \ldots, n\}$

Exercise: identify the kernel

gamma kernel: $x^{\alpha-1} e^{-\beta x}$

Exercise.

$$\int_0^\infty x^{\alpha-1} e^{-\beta x} \, dx = ?$$

$$\frac{\Gamma(\alpha)}{\beta^\alpha}$$

Law of total expectation

$$E \underbrace{E(X|\theta)}_{g(\theta)}$$

$$= \int \left[\int x \, p(x|\theta) \, dx \right] p(\theta) \, d\theta$$

$$= \int x \int \underbrace{p(x|\theta) \, p(\theta)} \, d\theta \, dx$$

$$= \int x \int p(x,\theta) \, d\theta \, dx$$

$= \int x \, p(x) \, dx$ by rule of marginal prob.

$= EX \quad \square$

Defn **exchangeable** (subscripts don't matter)

Let $p(y_1, \ldots, y_n)$ be the joint density of $Y_1, \ldots, Y_n$. If $p(y_1, \ldots, y_n) = p(y_{\pi_1}, \ldots, y_{\pi_n})$ for all permutations π of $\{1, \ldots, n\}$ then $Y_1, \ldots, Y_n$ are **exchangeable**.

Ex 1: Urn with 2 red, 1 green

$$p(Y_1 = \text{red}, Y_2 = \text{green}) = p(Y_1 = \text{red}) \cdot p(Y_2 = \text{green} \mid Y_1 = \text{red})$$
$$= 2/3 \cdot 1/2$$
$$= 2/6 = 1/3$$

$$p(Y_1 = \text{green}, Y_2 = \text{red}) = p(Y_1 = \text{green}) \cdot p(Y_2 = \text{red} \mid Y_1 = \text{green})$$
$$= 1/3 \cdot 1$$
$$= 1/3$$

Y_1, Y_2 **are** exchangeable even though not independent.

Ex 2:

coin 1 is a fair coin

coin 2 is double sided (heads only)

$Pr(Y_1 = H) = 0.5$

$Pr(Y_2 = H) = 1$

$p(0, 1) = 0.5$

$p(1, 0) = 0$

Y_1, Y_2 **are not** exchangeable.

Claim:

If $\theta \sim p(\theta)$ and $Y_1, \ldots, Y_n$ are conditionally iid given θ, then marginally (unconditional on θ) $Y_1, \ldots, Y_n$ are exchangeable.

Proof:

$$p(y_1, \ldots, y_n) = \int p(y_1, \ldots, y_n | \theta)\, p(\theta)\, d\theta$$ by rule of marginal prob.

$$= \int \left\{ \prod_{i=1}^{n} p(y_i | \theta) \right\} p(\theta)\, d\theta$$ by cond'l iid

$$= \int \left\{ \prod_{i=1}^{n} p(y_{\pi_i} | \theta) \right\} p(\theta)\, d\theta$$ products commute

$$= p(y_{\pi_1}, \ldots, y_{\pi_n})$$

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de Finetti's thm:

exchangeable $Y_1, \ldots, Y_n$ $\forall n$

$\Rightarrow$ $Y_1, \ldots, Y_n | \theta$ iid (for some parameter θ) and prior distribution $p(\theta)$.

- very cool because exchangeability is common!

$Y_1, \ldots, Y_n \to$ from repeatable experiment
$\to$ sampled w/ replacement
$\to$ ∞ pop'n w/o replacement

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$$X_i = \begin{cases} 1 & (H) \\ 0 & (T) \end{cases}$$

$X_1 \perp X_2$?

$P(X_{100} = 1 \mid X_1 = 0, X_2 = 0, \ldots\ldots, X_{99} = 0)$ $> .5$
$< .5$
$= .5$?

So $X_i \not\perp X_j$

But exchangeable seems plausible.

$\Rightarrow$ $X_i \perp X_j \mid \theta$ & identically distributed.

therefore $p(X_1, \ldots, X_n \mid \theta) = p(X_1 \mid \theta)\, p(X_2 \mid \theta) \cdots p(X_n \mid \theta)$.

What could θ be here?

Ex: $p(X_i = 1 \mid \theta) = \theta$

$p(X_i = 0 \mid \theta) = (1-\theta)$

Together, $p(X_i = x_i \mid \theta) = \theta^{x_i}(1-\theta)^{1-x_i}$

Exercise: write the joint density:

$p(X_1, \ldots, X_n \mid \theta)$

Sol'n: $\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$

This is called: (1) the joint density of the data

(2) "the data generative model"

(3) the <u>likelihood</u> function

the eval of the likelihood depends on the sum.